B_c-meson sum rules at next-to-leading order

Andrei I. Onishchenko

Institute for Theoretical and Experimental Physics,
B. Cheremushkinskaja 25, Moscow, 117259 Russia

Abstract

We present B_c-meson two-point sum rules at next-to-leading order in NRQCD approximation. Analitycal results for perturbative spectral density and gluon condensate contribution with account for summed Coulomb corrections are given. Estimates of various c-quark masses together with couplings and masses of lowest lying B_c-meson resonances are performed.

1 Introduction

The study of the physics of B_c-meson[1] has already a long story. The collaborative efforts of many scientists around the world have led to remarkable predictions, describing spectroscopy [1, 2], decays [3, 4, 5, 6, 7] and production mechanisms [8] of this object. The theoretical work done was very helpful in the experimental search and discovery of this meson by CDF collaboration [9]. As it was many times earlier with other particles, the observation of B_c-meson in nature have not diminished the interest of physicists to this object. People wonder how much we can learn from this meson about the Standard Model (SM) of particle interactions. In this paper we explore a potential of B_c - meson in extracting the parameters of SM lagragian, namely, the heavy quark masses.

What concerns its spectroscopic properties, this meson stands among the families of charmonium $\bar{c}c$ and bottominum $\bar{b}b$: two heavy quarks move nonrelativistically, since the confinement scale, determining the presence of light degrees of freedom (sea of gluons and quarks), is suppressed with respect to the heavy quark masses m_Q as well as the Coulomb-like exchanges result in transfers about $\alpha_s m_Q^2$, which is again much less than the heavy quark mass. It is precisely the nonrelativistic nature of heavy quark dynamics in the B_c-meson, that offers us a possibility to use the NRQCD framework and gain more inside on QCD dynamics of the constituent heavy quarks. Recently, within this framework the NLO and NNLO NRQCD sum rules were derived and analyzed for the case of Υ-family [11]. The result of this analysis was a precise determination of pole, running, 1S, potential subtracted and kinetic b-quark masses. Here we employ the same NRQCD sum rule method to determine the numerical values for different definitions of c-quark masses together with couplings and masses of lowest lying B_c-meson resonances. The present work is similar to the analysis performed for Υ-family with differences in particular analytical expressions for theoretical moments of two-point correlator.

The paper is organized as follows. In section 2 we remind the reader the QCD sum rule framework for the two-point sum rule. Section 3 introduces the NRQCD framework and we comment on its connection with QCD one. Section 4 explains the strategy used for calculation of NRQCD two-point correlation function. In section 5 we calculate the perturbative theoretical expressions for moments of correlation function. Section 5 deals with the corrections to the correlator, coming from the gluon condensate operator. In section 7 we present our results on the different c-quark mass definitions, B_c-meson mass and coupling constant. The detail discussion on optimization methods is also given. And finally, section 8 contains our summary.

[1]For review see [10].

2 Two-point sum rules

In the QCD sum rules approach[12, 13] meson bound states are described by local interpolating currents of the form $J = \bar{q}_1 \Gamma q_2$, where Γ is a suitable Dirac structure to account for meson quantum numbers. Thus, the studied object in QCD sum rules is a coupling of meson under consideration to corresponding current. In the case of B_c meson this coupling is defined by the following equation

$$\langle 0|\bar{b} i\gamma_5 c|B_c\rangle = \frac{f_{B_c} M_{B_c}^2}{m_b + m_c} \tag{1}$$

The estimates of B_c-meson structure constant in QCD sum rule framework were already performed in a number of papers [14]. The distinctive feature of the present work is a complete next-to-leading order analysis, containing correct treatment of Coulomb corrections. For discussion of importance of such corrections see, for example [7].

In QCD sum rules framework, the B_c-meson structure constant is naturally obtained from two-point correlator of the following form

$$\Pi(q^2) = i \int d^4x e^{iq\cdot x} \langle 0|J_{B_c}(x) J_{B_c}(0)^+|0\rangle, \tag{2}$$

where $J_{B_c} = \bar{b} i\gamma_5 c$.

The left hand side of Eq.(2) can be computed in QCD, for $|q^2|$ much larger than Λ^2_{QCD} (or , what alternatively, for $|q^2 - (m_c + m_b)^2|$ much larger than Λ^2_{QCD}), with the use of short-distance Operator Product Expansion (OPE) for correlation function under consideration.

$$\Pi(q^2)_{QCD} = \Pi_{pert}(q^2) + C_{G^2}(q^2)\langle\frac{\alpha_s}{\pi}G^2\rangle + \ldots \quad , \tag{3}$$

where $C_{G^2}(q^2)$ is a Wilson coefficient of gluon condensate operator and dots in right hand side of Eq.(3) present contributions of operators with higher dimension ($d > 4$).

The connection to physical spectrum of B_c-meson can be obtained by writing the following dispersion relation

$$\Pi(q^2) = \frac{1}{\pi}\int \frac{\rho(s)_{had} ds}{s - q^2} + \text{subtractions}, \tag{4}$$

where

$$\rho(s)_{had} = \pi \frac{f_{B_c}^2 M_{B_c}^4}{(m_b + m_c)^2}\delta(s - M_{B_c}^2) + \rho(s)_{pert}\theta(s - s_{thr}), \tag{5}$$

Here we have taken into account only lowest lying B_c-meson state, s_{thr} is a continuum threshold and $\rho(s)_{pert}$ is connected to $\Pi_{pert}(q^2)$ via the following dispersion relation

$$\Pi_{pert}(q^2) = \frac{1}{\pi}\int \frac{\rho(s)_{pert} ds}{s - q^2} + \text{subtractions} \tag{6}$$

In the numerical analysis we however explore a different anzaz, on which we will comment in the section with numerical results.

There are several schemes of QCD sum rules, which can be used for the extraction of quantities, you are interesting in. The most popular among them are momentum and Borel schemes. In this paper we will employ the first one and the studied object will be the momentum of two-point correlation function, given by the following expression

$$P_n = n!\left(\frac{d}{dq^2}\right)^n \Pi(q^2)|_{q^2=0} \tag{7}$$

Thus far we have discussed QCD sum rule framework for determination of B_c-meson structure constant. As a primary goal of this paper is to perform a consistent analysis of the same quantity in NRQCD approximation, in the next section we define NRQCD sum rule framework and comment on connection of the latter with QCD sum rule analysis.

3 NRQCD approximation

In this section we set up a consistent NRQCD framework, in which the two-point correlation function $\Pi(q^2)$ can be determined in a systematic manner at next-to-leading order. Our presentation in this and the next sections closely follow that of [29], so we advice the reader to read that paper for more detail.

NRQCD is an effective field theory of QCD designed to handle nonrelativistic heavy-quark-antiquark systems to in principle arbitrary precision. Considering all quarks of the first and second generation as massless the NRQCD Lagrangian reads [15]

$$\begin{aligned}
\mathcal{L}_{\text{NRQCD}} &= -\frac{1}{2}\,\mathrm{Tr}\,G^{\mu\nu}G_{\mu\nu} + \sum_{q=u,d,s,c}\bar{q}\,i\not{D}\,q \\
&+\psi^\dagger\left[iD_t + a_1\,\frac{\boldsymbol{D}^2}{2\,M_t} + a_2\,\frac{\boldsymbol{D}^4}{8\,M_t^3}\right]\psi + \ldots \\
&+\psi^\dagger\left[\frac{a_3\,g}{2\,M_t}\,\boldsymbol{\sigma}\cdot\boldsymbol{B} + \frac{a_4\,g}{8\,M_t^2}\,(\,\boldsymbol{D}\cdot\boldsymbol{E} - \boldsymbol{E}\cdot\boldsymbol{D}\,) + \frac{a_5\,g}{8\,M_t^2}\,i\,\boldsymbol{\sigma}\,(\,\boldsymbol{D}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{D}\,)\right]\psi + \ldots \\
&+\chi\chi^\dagger \text{ bilinear terms and higher dimensional operators}\,.
\end{aligned} \tag{8}$$

The gluons and massless quarks are described by the conventional relativistic Lagrangian, where $G_{\mu\nu}$ is the gluon field strength tensor, q the Dirac spinor of a massless quark and D_μ the gauge covariant derivative. For convenience, all color indices in Eq. (8) and throughout this work are suppressed. The nonrelativistic c and $\bar{b}$ quarks are described by the Pauli spinors ψ and χ, respectively. D_t and $\boldsymbol{D}$ are the time and space components of the gauge covariant derivative D and $E^i = G^{0i}$ and $B^i = \frac{1}{2}\epsilon^{ijk}G^{jk}$ the electric and magnetic components of the gluon field strength tensor (in Coulomb gauge). The straightforward $\chi^\dagger\chi$ bilinear terms are omitted and can be readily obtained. The short-distance coefficients $a_1, \ldots, a_5$ are normalized to one at the Born level. The actual form of the higher order contributions to the short-distance coefficients $a_1, \ldots, a_5$ is irrelevant for this work, as we will later use the "direct matching" procedure at the level of the final result for the correlation function.

Now let us discuss our correlation function in NRQCD approximation. Eq.(2) can be rewritten as

$$\Pi(q^2) = i\langle 0|T\tilde{J}_{B_c}(q)\tilde{J}_{B_c}(-q)^+|0\rangle, \tag{9}$$

where $\tilde{J}_{B_c}(q) = (\bar{b}i\gamma_5 c)(q)$. Expressing Dirac fields for $\bar{b}$ and c - quarks in terms of Pauli spinors χ and ψ

$$u_c(\mathbf{q}) \;=\; \sqrt{\frac{E_c+m_c}{2E_c}}\begin{pmatrix}\psi \\ \frac{\mathbf{q}\cdot\boldsymbol{\sigma}}{E_c+m_c}\psi\end{pmatrix}, \tag{10}$$

$$v_b(-\mathbf{q}) \;=\; \sqrt{\frac{E_b+m_b}{2E_b}}\begin{pmatrix} \frac{(-\mathbf{q})\cdot\boldsymbol{\sigma}}{E_b+m_b}\chi \\ \chi \end{pmatrix}, \tag{11}$$

we have[2]

$$\tilde{J}_{B_c}(q) \;\approx\; -i\left((\chi^\dagger\psi)(q) \;-\; \frac{1}{8}\left(\frac{m_b-m_c}{m_b m_c}\right)^2 ((\mathbf{D}\chi)^\dagger\cdot\mathbf{D}\psi)(q) \;+\; \ldots\right) \tag{12}$$

$$\tilde{J}_{B_c}(-q)^\dagger \;\approx\; -i\left((\psi^\dagger\chi)(-q) \;-\; \frac{1}{8}\left(\frac{m_b-m_c}{m_b m_c}\right)^2 ((\mathbf{D}\psi)^\dagger\cdot\mathbf{D}\chi)(-q) \;+\; \ldots\right) \tag{13}$$

Inserting these expansions into Eq. (9) we obtain

$$\begin{aligned} i\Pi(q^2) \;=\; & C_1(\mu_{\rm hard},\mu_{\rm fact})\mathcal{A}_1(E,\mu_{\rm soft},\mu_{\rm fact}) \\ & -\frac{1}{4}\left(\frac{m_b-m_c}{m_b m_c}\right)^2 C_2(\mu_{\rm hard},\mu_{\rm fact})\mathcal{A}_2(E,\mu_{\rm soft},\mu_{\rm fact}) + \ldots, \end{aligned} \tag{14}$$

where

$$\mathcal{A}_1 \;=\; \langle 0|(\chi^\dagger\psi)(\psi^\dagger\chi)|0\rangle, \tag{15}$$

$$\mathcal{A}_2 \;=\; \frac{1}{2}\langle 0|(\chi^\dagger\psi)((\mathbf{D}\psi)^\dagger\cdot\mathbf{D}\chi) + \text{h.c.}|0\rangle, \tag{16}$$

The right-hand side of Eq. (14) just represents an application of the factorization formalism proposed in [15]. The second term in this expression is suppressed by v^2, i.e. of next-to-next-to-leading order and thus of no relevance to us in present analysis. The nonrelativistic current correlator $\mathcal{A}_1$ contains the resummation of the singular Coulomb terms. It incorporates all the long-distance, dynamics governed by soft scales like the relative three momentum $\sim m_{red}v$ or the binding energy of the $c\bar{b}$ system $\sim m_{red}v^2$. The constant C_1 (it is normalized to one at the Born level), on the other hand, describes short-distance effects involving hard scales of the order of heavy quark mass. It is represented only by a simple power series in α_s and does not contain any resummations in α_s. We would also like to note, that C_1 is independent of q^2. In Eq. (14) we have also indicated the dependence of the NRQCD correlators and the short-distance coefficients on the various renormalization scales: The factorization scale $\mu_{\rm fact}$ essentially represents the boundary between hard and soft momenta. The dependence on the factorization scale becomes explicit because of ultraviolet (UV) divergences contained in NRQCD. Because, as in any effective field theory, this boundary is not defined unambiguously, both the correlators and the short-distance coefficients in general depend on $\mu_{\rm fact}$. The soft scale $\mu_{\rm soft}$ and the hard scale $\mu_{\rm hard}$, on the other hand, are inherent to the correlators and the short-distance constants, respectively, governing their perturbative expansion. If we would have all orders in α_s and v at hand, the dependence of correlation function $\Pi(q^2)$ on variations of each the three scales would vanish exactly. Unfortunately, we only perform the calculation up to next-to-leading order in α_s and v which leads to a residual dependence[3] on the three scales $\mu_{\rm fact}$, $\mu_{\rm soft}$ and $\mu_{\rm hard}$.

4 Calculation of NRQCD correlator

To calculate the NRQCD correlator $\mathcal{A}_1$ we use methods originally developed for QED bound state calculations in the framework of NRQED [16, 17, 18, 19] and apply them to B_c-meson bound state

[2] Here and later in the paper, except stated otherwise, by m_c and m_b we mean the pole heavy quark masses.

[3] Here we would like to note, that the object studied P_n at NLO does not depend on μ_{fact}

description in the kinematic regime close to the threshold. At next-to-leading order quarks, composing B_c-meson experience only instantaneous interactions, given by the following potentials

$$
\begin{aligned}
V_c^{(0)}(\vec{r}) &= -\frac{C_F\alpha_s}{r}, && (17)\\
V_c^{(1)}(\vec{r}) &= V_c^{(0)}\left(\frac{\alpha_s}{4\pi}\right)[2\beta_0\ln(\tilde{\mu}r)+a_1], \quad \tilde{\mu}\equiv e^{\gamma_E}\mu_{\rm soft}, && (18)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_0 &= \frac{11}{3}C_A-\frac{4}{3}Tn_l,\\
a_1 &= \frac{31}{9}C_A-\frac{20}{9}Tn_l,\\
n_l &= 3.
\end{aligned}
\qquad (19)
$$

Here $r\equiv|\vec{r}|$, $C_F=\frac{4}{3}$, $C_A=3$, $T=\frac{1}{2}$, $\alpha_s\equiv\alpha_s(\mu_{\rm soft})$ and γ_E is the Euler-Mascheroni constant.

Thus, we can conclude that the problem of B_c-meson description close to threshold can be treated as a pure quantum two-body problem, so that we can use the well known analytic solutions of the non-relativistic Coulomb problem for positronium [20, 21, 22] and Rayleigh-Schrödinger time-independent perturbation theory (TIPT) to determine the corrections caused by all higher order interactions and effects.

The calulational procedure for two-point NRQCD correlation function may be devided in the following two steps

Step 1: *Solution of the Schrödinger equation.* – The Green function of the next-to leading Schrödinger equation is calculated incorporating the potentials displayed above. The correlator $\mathcal{A}_1$ is directly related to the zero-distance Green function of the Schrödinger equation.

Step 2: *Matching calculation.* – The short-distance constant C_1 is determined at $\mathcal{O}(\alpha_s)$ by matching QCD current J_{B_c} to corresponding NRQCD one.

4.1 Solution of the Schrödinger equation

The nonrelativistic correlator $\mathcal{A}_1$ is calculated by determining the Green function of the Schrödinger equation ($E\equiv\sqrt{q^2}-(m_b+mc)$)

$$
\left(-\frac{\vec{\nabla}^2}{2m_{red}}+\left[V_c^{(0)}(\vec{r})+V_c^{(1)}(\vec{r})\right]-E\right)G(\vec{r},\vec{r}',E)=\delta^{(3)}(\vec{r}-\vec{r}') \qquad (20)
$$

The relation between the correlator $\mathcal{A}_1$ and Green function reads

$$
\mathcal{A}_1 = 6\left[\lim_{|\vec{r}|,|\vec{r}'|\to0}G(\vec{r},\vec{r}',E)\right]. \qquad (21)
$$

Eq. (21) can be quickly derived from the facts that $G(\vec{r},\vec{r}',\tilde{E})$ describes the propagation of $\bar{b}$ and c quark pair, which is produced and annihilated at relative distances $|\vec{r}|$ and $|\vec{r}'|$, respectively, and that the same quark pair is produced and annihilated through the J_{B_c} current at zero distances. Therefore $\mathcal{A}_1$ must be proportional to $\lim_{|\vec{r}|,|\vec{r}'|\to0}G(\vec{r},\vec{r}',E)$. The correct proportionality constant can then be determined by matching the result of full QCD for perturbative spectral density , presented in Appendix A, to the imaginary part of the Green function of the Coulomb nonrelativistic Schrödinger equation. We would like to emphasize that the zero-distance Green function on the right hand side of Eqs. (21) contains UV divergences which have to regularized. In the actual calculations we impose the explicit short-distance cutoff $\mu_{\rm fact}$. As mentioned before, this is the reason why the correlators and the short-distance constants depend explicitly on the (factorization) scale $\mu_{\rm fact}$. In this work we solve equation (20) perturbatively by starting from well known Green function $G_c^{(0)}$ of the nonrelativistic Coulomb problem [20, 21, 22]

$$\Big(- \frac{\nabla^2}{m_{red}} - V_c^{(0)}(\vec{r}) - E \Big) G_c(\vec{r}, \vec{r}', E) \,=\, \delta^{(3)}(\vec{r} - \vec{r}') \tag{22}$$

and incorporate all the higher order terms using TIPT.

The most general form of the Coulomb Green function reads ($r \equiv |\vec{r}|$, $r' \equiv |\vec{r}'|$)

$$\begin{aligned} G_c^{(0)}(\vec{r}, \vec{r}', E) \;&=\; - \frac{m_{red}}{4\,\pi\,\Gamma(1+i\,\rho)\,\Gamma(1-i\,\rho)} \int\limits_0^1 dt \int\limits_1^\infty ds \Big[s\,(1-t) \Big]^{i\,\rho} \Big[t\,(s-1) \Big]^{-i\,\rho} \times \\ &\times \frac{\partial^2}{\partial t\,\partial s} \Bigg[\frac{t\,s}{|\,s\,\vec{r} - t\,\vec{r}'\,|} \, \exp\Big\{ i\,p \Big(|\,\vec{r}'\,|\,(1-t) + |\,\vec{r}\,|\,(s-1) + |\,s\,\vec{r} - t\,\vec{r}'\,| \Big) \Big\} \Bigg] , \quad r' < r , \end{aligned} \tag{23}$$

where

$$p \,\equiv\, m_{red}\,v \,=\, \sqrt{m_{red}\,(E + i\,\epsilon)}\,, \qquad \rho \,\equiv\, \frac{C_F\,a_s}{2\,v} \tag{24}$$

and Γ is the gamma function. The case $r < r'$ is obtained by interchanging r and r'. $G_c^{(0)}(\vec{r}, \vec{r}', E)$ represents the analytical expression for the sum of ladder diagrams with Coulomb exchanges. The analytic form of the Coulomb Green function shown in Eq. (23) has been taken from Ref. [20]. Fortunately we do not need the Coulomb Green function in its most general form but only its S-wave component with one of the relative quark distances set to zero.[4]

$$\begin{aligned} G_c^{(0)}(0, r, E) \;&=\; G_c^{(0)}(0, \vec{r}, E) \,=\, -\,i\,\frac{m_{red}\,p}{2\,\pi}\,e^{i\,p\,r} \int\limits_1^\infty dt\, e^{2\,i\,p\,r\,t} \Big(\frac{1+t}{t} \Big)^{i\,\rho} \\ &=\; -\,i\,\frac{m_{red}\,p}{2\,\pi}\,e^{i\,p\,r}\,\Gamma(1-i\,\rho)\,U(1-i\,\rho, 2, -2\,i\,p\,r) \end{aligned} \tag{25}$$

where $U(a, b, z)$ is a confluent hypergeometric function [24, 25]. It is an important fact that $G_c^{(0)}(0, \vec{r}, E)$ diverges for the limit $r \to 0$ because it contains power ($\propto 1/r$) and logarithmic ($\propto \ln r$) divergences [26]. As we have explained before these ultraviolet (UV) divergences are regularized by imposing the small distance cutoff $\mu_{\rm fact}$. The regularized form of $\lim_{r\to 0} G_c(0, \vec{r}, E)$ reads

$$G_c^{(0),\,reg}(0, 0, E) \,=\, \frac{m_{red}^2}{4\,\pi} \Big\{ i\,v - C_F\,a_s \Big[\ln(-i\frac{m_{red}\,v}{\mu_{\rm fact}}) + \gamma_{\rm E} + \Psi\Big(1 - i\,\frac{C_F\,a_s}{2v} \Big) \Big] \Big\} , \tag{26}$$

where the superscript "reg" indicates the cutoff regularization and $\Psi(z) = d \ln \Gamma(z)/dz$ is the digamma function. For the regularization we use the convention where all power divergences $\propto \mu_{\rm fact}$ are freely

[4] In the section, discussing the nonperturbative corrections, coming from gluon condensate operator, one more representation of Coulomb Green function will be introduced.

dropped and only logarithmic divergences $\propto \ln(\mu_{\rm fact}/m_{red})$ are kept. Further, we define $\mu_{\rm fact}$ such that in the expression between the brackets all constants except the Euler-Mascheroni constant $\gamma_{\rm E}$ are absorbed. The results for any other regularization scheme which suppressed power divergences (like the $\overline{\rm MS}$ scheme) can be obtained by redefinition of the factorization scale. For convenience we suppress the superscript "reg" from now in this work.

The Coulomb Green function contains $c\bar{b}$ bound state poles at the energies $\sqrt{s}_n = m_b + m_c - C_F^2 a_s^2 m_{red}/4n^2$ ($n = 1, 2, \ldots \infty$). These poles come from the digamma function in Eq. (26) and correspond to the nonrelativistic positronium state poles known from QED [27]. They are located entirely *below* the threshold point $\sqrt{s}_{\rm thr} = m_b + m_c$. This can be seen explicitly from the expression for imaginary part of Coulomb Green function $G_c^{(0)}(0,0,E)$

$$\begin{aligned} \mathrm{Im}\Big[\, G_c^{(0)}(0,0,E)\,\Big] &= 4\,\pi\, m_{red} \sum_{n=1}^{\infty} |\Psi_n(0)|^2\, \delta(s - s_n) \; + \\ & \Theta(E)\, \frac{1}{4\pi}\, m_{red}^2\, \frac{C_F\, a_s\, \pi}{1 - \exp(-\frac{C_F\, a_s\, \pi}{v})}\,, \end{aligned} \tag{27}$$

where $|\Psi_n(0)|^2 = (m_{red} C_F a_s)^3/8\pi n^3$ is the modulus squared of the LO nonrelativistic bound state wave functions for the radial quantum number n. The continuum contribution on the right-hand side of Eq. (27) is sometimes called "Sommerfeld factor" or "Fermi factor" in the literature. And the second term from the first line in Eq. (27) describes the resonance contributions. And finally, the corrections to the zero-distance Coulomb Green function calculated below lead to higher order contributions to the bound state energy levels poles and the continuum.

Let us now come to the determination of the corrections to the zero-distance Coulomb Green function coming from the remaining term in the Schrödinger equation (20). At next-to-leading order only the one-loop contributions to the Coulomb potential, $V_c^{(1)}$ have to considered. Using first order TIPT in configuration space representation the NLO corrections to $G_c^{(0)}(0,0,E)$ reads

$$G_c^{(1)}(0,0,E) \;=\; -\int d^3\vec{r}\, G_c^{(0)}(0,r,E)\, V_c^{(1)}(\vec{r})\, G_c^{(0)}(r,0,E)\,. \tag{28}$$

We will not calculate explicitly NLO correction to the Coulomb Green function here as the goal of this paper is to calculate the theoretical expressions for moments. The later can be most conveniently calculated by dispersion integration, using the following representation for the theoretical moments

$$P_n^{th} = \frac{6}{\pi}\int_0^\infty \frac{ds}{s^n} \mathrm{Im}\{C_1(\mu_{hard},\mu_{soft}) G_c(0,0,E)\}, \tag{29}$$

where $E = \sqrt{s} - m_b - m_c$.

4.2 Determination of the short distance coefficients

The short-distance coefficient C_1 and C_2 can be determined by matching perturbative calculations of the matrix elements in full QCD and NRQCD. A convenient choice for matching is the matrix element between the vacuum and the state $|c\bar{b}\rangle$ consisting of a c and a $\bar{b}$ on their perturbative mass shells with nonrelativistic four-momenta p and p' in the center of momentum frame: $\mathbf{p} + \mathbf{p}' = 0$. The matching condition is

$$\langle 0|\bar{b}\gamma^0\gamma_5 c|c\bar{b}\rangle\Big|_{\rm QCD} = C_1\, \langle 0|\chi_b^\dagger \psi_c|c\bar{b}\rangle\Big|_{\rm NRQCD} +\; C_2\, \langle 0|(\mathbf{D}\chi_b)^\dagger\cdot \mathbf{D}\psi_c|c\bar{b}\rangle\Big|_{\rm NRQCD} +\; \ldots, \tag{30}$$

To determine the short distance coefficients to order α_s, we must calculate the matrix elements on both sides of (30) to order α_s. It is sufficient to calculate the order-α_s correction only for the coefficient C_1, since as we already said the contribution proportional to C_2 is suppressed by v^2. The coefficient C_1 can be isolated by taking the limit $\mathbf{p} \to 0$, in which case the matrix element of $(\mathbf{D}\chi_b)^\dagger \cdot \mathbf{D}\psi_c$ vanishes. Such calculations were done in [28], where the following result was obtained for C_1:

$$C_1 \;=\; 1 \;+\; \frac{\alpha_s(m_{\rm red})}{\pi}\left[\frac{m_b - m_c}{m_b + m_c}\log\frac{m_b}{m_c} - 2\right], \tag{31}$$

where $m_{\rm red}$ is the scale of the running coupling constant. The same result gives the direct matching of NRQCD correlator with QCD one, taking into account factor 2 for α_s correction.

5 Dispersion integration

In this section we will discuss issues related to dispersion integration in expressions for NRQCD moments. In general, the integration of spectral density over complete covariant form of integration measure $\frac{ds}{s^{n+1}}$ is quite cumbersome However, in NRQCD approximation, as we will see soon, this task significantly simplifies. Let us make a change of variables $E = \sqrt{s} - m_b - m_c$ and consider a limit $E \ll m_b + m_c$. In this case the integration measure takes the form

$$\begin{aligned} \frac{ds}{s^{n+1}} &= \frac{1}{(m_b+m_c)^{2n}}\frac{2dE}{m_b+m_c}\exp\{-(2n+1)\ln(1+\frac{E}{m_b+m_c})\} \\ &\approx \frac{1}{(m_b+m_c)^{2n}}\frac{2dE}{m_b+m_c}\exp\{\frac{2En}{m_b+m_c}\} + O(\frac{2E}{m_b+m_c}) \end{aligned} \tag{32}$$

The dispersion integration for NRQCD moments in this limit is

$$P_n^{th} = \frac{1}{(m_b+m_c)^{2n}}\int_{E_{bind}}\frac{2dE}{m_b+m_c}\exp\{-\frac{2En}{m_b+m_c}\}R^{th}_{NLO}(E), \tag{33}$$

where $E_{bind} = \frac{m_{red}C_F^2\alpha_s^2}{2}$ is the negative binding energy of the lowest lying resonance. This integration is performed most efficiently by deforming the path of integration into negative complex energy plane, such that the part of the integration path parallel to imaginary axis is far away from bound state poles. That is

$$P_n^{th} = \frac{1}{(m_b+m_c)^{2n}}\frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{2dE}{m_b+m_c}\exp\{-\frac{2En}{m_b+m_c}\}C_1A_1(E), \tag{34}$$

where $\gamma \ll E_{bind}$. Note, that in the above equation the real part of correlator A_1 is also present, which is needed for integration over the new path. After performing the second change of variables $E \to -\tilde{E}$

$$P_n^{th} = \frac{\pi}{(m_b+m_c)^{2n}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{2d\tilde{E}}{m_b+m_c}\exp\{\frac{2\tilde{E}n}{m_b+m_c}\}C_1A_1(-\tilde{E}) \tag{35}$$

we see, that the problem of evaluation of NRQCD moments can be related to the inverse Laplace transform of integrand expression, for with there are a lot of tables in literature. The procedure of taking inverse Laplace transform can be further simplified by noting that integration path is far away from bound state energies and hence the integrand can be safely expanded in α_s.

Following the steps, described above, and using relations for inverse Laplace transform from Appendix B the NRQCD moments in leading order of NRQCD expansion have the form

$$[P_n^{th}]^{LO} = \frac{3(m_{red})^{3/2}(m_b+m_c)^{1/2}}{2\sqrt{\pi}(m_b+m_c)^{2n}n^{3/2}}\Phi^0(\gamma), \tag{36}$$

where

$$\Phi^0(\gamma) = 1 + 2\sqrt{(\pi)}\gamma + \frac{2\pi^2}{3}\gamma^2 + 4\sqrt{\pi}\sum_{p=1}^{\infty}\left(\frac{\gamma}{p}\right)^3 \exp\{\left(\frac{\gamma}{p}\right)^2\}[1+\mathrm{erf}\left(\frac{\gamma}{p}\right)] \tag{37}$$

and

$$\gamma \equiv \frac{C_F\alpha_s m_{red}^{1/2} n^{1/2}}{(m_b+m_c)^{1/2}} \tag{38}$$

The calculation of NLO order correction to moments, coming from correction to potential (18) is a bit more involved. With the help of presentation (25) for Coulomb Green function we have

$$\begin{aligned}
[P_n^{th}]^{NLO} &= \frac{6}{(m_b+m_c)^{2n}}\frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{2d\tilde{E}}{m_b+m_c}\exp\{\frac{2\tilde{E}n}{m_b+m_c}\}\times \\
&\left\{4\pi\int_0^\infty r^2dr\left(\frac{m_{red}k}{\pi}\right)^2\int_0^\infty dt\int_0^\infty due^{-2kr(t+u+1)}\left(\frac{(1+t)(1+u)}{tu}\right)^\lambda\right\}\times \\
&C_F\left(\frac{\alpha_s^2}{4\pi}\right)\left\{2\beta_0(\frac{1}{r}\log(\tilde{\mu}r)) + \frac{a_1}{r}\right\},
\end{aligned} \tag{39}$$

where $\lambda = \frac{C_F m_{red}\alpha_s}{k}$ and $k = 2m_{red}\tilde{E}$. The integration over r can be easily performed explicitly, while the situation with integrations over t and u is far more complicated. However, as we noted above the integrand expression in the case under consideration can be expanded in series over λ[5] . Then the result of integration over all terms in λ expansion, except λ^0 can easily expressed in terms of functions w_p^1 and w_p^0, introduced in [29][6]. As for the term with λ^0, which contain a manifestly divergent integral, it can be easily calculated in terms of $K(\tau) = \langle 0|\exp(-H\tau)|0\rangle$, using the free evolution function $K_0(\tau) = \langle 0|\exp(-H_0\tau)|0\rangle$. Here $H = H_0 - \frac{C_F\alpha_s}{r}$, $H_0 = -\frac{\vec{\nabla}^2}{2m_{red}}$ and τ is an Euclidean time. The evolution function $K(\tau)$ can be related to the n'th NRQCD moment by the following relation

$$P_n^{th} = \frac{6}{(m_b+m_c)^{2n}}\frac{2\pi}{m_b+m_c}K(\frac{2n}{m_b+m_c}) \tag{40}$$

Using this relation and explicit expression for free quark propagation function:

$$\langle\mathbf{x}|\exp(-H_0\tau)|\mathbf{y}\rangle = \left(\frac{m_{red}}{2\pi\tau}\right)^{3/2}\exp\left(-\frac{m_{red}}{2\tau}(\mathbf{x}-\mathbf{y})^{\mathbf{2}}\right), \tag{41}$$

we can now easily evaluate the contribution of λ^0 term for NRQCD moments. The combined result for NLO order correction to NRQCD moments reads

$$[P_n^{th}]^{NLO} = \frac{3(m_{red})^{3/2}(m_b+m_c)^{1/2}}{2\sqrt{\pi}(m_b+m_c)^{2n}n^{3/2}}\Phi^1(\gamma), \tag{42}$$

[5] It's the same as the expansion in α_s

[6] The expressions for them can be found in Appendix B

where

$$\Phi^1(\gamma) = \frac{2\beta_0\alpha_s}{\sqrt{\pi}}\gamma\left\{\frac{1}{2}\log(\frac{\mu_1 e^{\gamma_E}\sqrt{n}}{2\sqrt{m_{red}(m_b+m_c)}}) + \sum_{p=1}^{\infty}\gamma^p[w_p^1 + w_p^0(\log\left(\frac{2m_{red}(m_b+m_c)}{\mu_1\sqrt{n}}\right) + \frac{1}{2}\Psi\left(\frac{p}{2}\right))]\right\} \quad (43)$$

Here $\mu_1 = \mu_{soft}\exp\left(\frac{a_1}{2\beta_0}\right)$. The full perturbative result for NRQCD moments with account of hard gluon corrections reads

$$[P_n^{th}]^{pert} = (1 + \frac{\alpha_s(m_{\rm red})}{\pi}\left[\frac{m_b - m_c}{m_b + m_c}\log\frac{m_b}{m_c} - 2\right])\{[P_n^{th}]^{LO} + [P_n^{th}]^{NLO}\} \quad (44)$$

6 Nonperturbative corrections

In this subsection we would like to consider corrections, given by gluon condesate operator. The calculation is very similar to the one done previously by Voloshin and Leutwyler [30] for the case of equal quark masses. For the determination of corrections to Coulomb Green function, coming from gluon condensate operator, we will exploit the fact, that the size of vacuum fluctuations of gluon field is much larger then the size of B_c-meson state. So, we can perform a multipole expansion for interaction of B_c-meson with gluon condensate, whose first term is

$$H_{\rm int} = -\frac{1}{2}g\xi^a\vec{r}\vec{E}^a, \quad (45)$$

where $\vec{r} = \vec{x}_c - \vec{x}_{\bar{b}}$, $g^2 = 4\pi\alpha_s$, $\xi^a = t_1^a - t_2^a$. By emploing colour and Lorenz invariance of the vacuum state one can relate the average over the vacuum state of two chromoelectric fields to the manifestly Lorenz-invariant value

$$\langle 0|E_i^a E_k^b|0\rangle = -\frac{1}{96}\delta_{ik}\delta^{ab}\langle 0|G_{\mu\nu}^c G_{\mu\nu}^c|0\rangle \quad (46)$$

Thus, the Coulomb Green function with account for gluon condesate corrections has the following form

$$\begin{aligned} G(\vec{x},\vec{y},E) \;&=\; G_{(0)} - \frac{1}{18}\langle 0|\pi\alpha_s G_{\mu\nu}^a G_{\mu\nu}^a|0\rangle \times \quad (47)\\ &\int\int d^3\vec{r}d^3\vec{r}'(\vec{r}\vec{r}')G_{(0)}(\vec{x},\vec{r},E)G_{(8)}(\vec{r},\vec{r}',E)G_{(0)}(\vec{r}',\vec{y},E), \end{aligned}$$

where $G_{(0)}$ and $G_{(8)}$ are Coulomb Green functions in singlet and octet colour states correspondingly. They are defined as solutions to the following equation

$$\left(\frac{p^2}{2m_{red}} + V_{(0,8)}(|\vec{x}|) + \frac{k^2}{2m_{red}}\right)G_{(0,8)}(\vec{x},\vec{y},-\frac{k^2}{2m_{red}}) = \delta(\vec{x}-\vec{y}), \quad (48)$$

where

$$V_0(r) = -\frac{\alpha^{(0)}}{r} = -\frac{4}{3}\frac{\alpha_s}{r}, \quad V_8(r) = -\frac{\alpha^{(8)}}{r} = \frac{2}{3}\frac{\alpha_s}{r}. \quad (49)$$

The expressions to NRQCD moments, coming form gluon condensate, can be calculated with the use of wave decomposition of NRQCD Green function

$$G_c^{(0,8)}(\vec{x},\vec{y};E) = \sum_{l=0}^{\infty}(2l+1)G_c^{(0,8)l}(x,y;E)P_l((\vec{x}\vec{y})/xy), \quad (50)$$

where

$$G_c^{(0,8)l}(x,y;-k^2/2m_{red}) = \frac{m_{red}k}{\pi}(2kx)^l(2ky)^l e^{k(x+y)}\sum_{s=0}^{\infty}\frac{L_s^{2l+1}(2kx)L_s^{2l+1}(2ky)s!}{(s+l+1-m_{red}\alpha^{(0,8)}/k)(s+l+1)!}, \tag{51}$$

where $x = |\vec{x}|$, $y = |\vec{y}|$. P_l and L_s^p are Legendre and Laguerre polynomials. The result for gluon condensate correction to Coulomb Green function is[7]:

$$[P_n^{th}]^{G^2} = -\frac{\sqrt{\pi}(m_{red})^{1/2}(m_b+m_c)^{1/2}n^{3/2}}{12(m_b+m_c)^{2n+3}}\chi(\gamma)\langle 0|\alpha_s G^a_{\mu\nu}G^a_{\mu\nu}|0\rangle \tag{52}$$

The function $\chi(\gamma)$ is given by the following equation

$$\begin{aligned}
\chi(\gamma) \;=\; & 4\sqrt{\pi}\{\sum_{p=1}^{\infty}(p+1)(p+2)(p+3)\{\frac{1}{9p+16}[\phi_2(\frac{\gamma}{p})-\phi_1(\frac{\gamma}{p})(1+\frac{p}{12}(25-\frac{6}{9p+16}))] \\
& +\frac{16}{9p+17}[\phi_2(\frac{\gamma}{p+1})-\phi_1(\frac{\gamma}{p+1})(1+\frac{p+1}{6}(5-\frac{3}{9n+17}))]+ \\
& \frac{4}{p+2}[\phi_2(\frac{\gamma}{p+2})-\frac{17}{18}\phi_1(\frac{\gamma}{p+2})]+\frac{16}{9p+19}[\phi_2(\frac{\gamma}{p+3})-\phi_1(\frac{\gamma}{p+3})(1- \\
& \frac{p+3}{6}(5+\frac{3}{9p+19}))]+\frac{1}{9p+20}[\phi_2(\frac{\gamma}{p+4})-\phi_1(\frac{\gamma}{p+4})(1-\frac{p+4}{12}(25+ \\
& \frac{6}{9p+20}))]\}+\frac{4}{81}\sum_{p=2}^{\infty}\frac{p^2-1}{(81p^2-1)(81p^2-4)}\phi_1(-\frac{\gamma}{8p})\},
\end{aligned} \tag{53}$$

where

$$\phi_1(x) = x^{-3}[e^{x^2}(1+\mathrm{erf}(x))-1]-(\frac{2}{\sqrt{\pi}})x^{-2}-x^{-1}, \tag{54}$$

$$\phi_2(x) = [e^{x^2}(1+\mathrm{erf}(x))-1]x^{-1}-\phi_1(x). \tag{55}$$

The above expression in the limit of equal quark masses coincides can be checked with the one derived previously by M.B.Voloshin [30]. However, this expression is quite complicated and in the range $\gamma \leq 1.5$ it is far more convenient to use an approximated formulum $\chi(\gamma) = e^{-0.80\gamma}\Phi^0(\gamma)$.

7 Numerical results

In this section we will discuss our numerical estimates for various c-quark mass definitions, the B_c-meson mass and coupling constant. Let us first discuss our anzaz for experimental spectral density, the need for which is dictated by our present lack of experimental measurements on higher B_c-meson states. The experience in potential models and quasi-local sum rules [31] allow us to write

$$\begin{aligned}
M_n &= M_1 + 2T\ln n, \quad M_1 = 6.3 \text{ GeV}, \quad T = 0.415 \text{ GeV}, \\
\frac{f_n^2}{f_l^2} &= \frac{1}{n}\frac{M_l}{M_n}, \qquad f_1(0^-) = 0.4 \text{ GeV},
\end{aligned} \tag{56}$$

[7] Here we assume that one should perform the dispertion integration in order to obtain the gluon condensate correction to NRQCD moments

where M_n is the mass of nS B_c-meson state, and f_n is its coupling. So that, the experimental spectral density has the form

$$\rho(s)_{hard} = \pi \sum_{n=1}^{3} \frac{f_n^2 M_n^4}{(m_b + m_c)^2} \delta(s - M_n^2) + \rho(s)_{pert} \theta(s - s_{thr}) \tag{57}$$

Now with the knowledge of experimental spectral density and thus of experimental moments we can compare them with theoretical ones. To begin with let's see what values of the pole c-quark mass will give us the required values of experimental moments. For the value of b-quark pole mass M_b^{pole}, needed for calculation of theoretical moments[8], we have taken the result of NNLO analysis for Υ family[11], so that we have $M_b^{pole} = 4.8 \pm 0.06$ GeV. Varying the soft scale in μ_{soft} the range $1.1 - 1.2$ GeV and fixing[9] the hard scale μ_{hard} to 2. GeV, from the stability of the ratios of experimental to theoretical moments we have that $M_c^{pole} = 1.7 - 2.1$ GeV. The range of pole c-quark mass obtained is quite large and to reduce it we need some extra condition to satisfy. Again, from the spectroscopy of B_c-meson family it is known that the characteristic value of strong coupling constant for heavy quark dynamics inside B_c-meson is in the range $\alpha_s = 0.43 - 0.48$. This extra requirement heavily constrains the value of c-quark pole mass, so that $M_c^{pole} = 2.03 \pm 0.06$ GeV. At Fig. 1 we have shown the ratios of experimental to theoretical moments as functions of momentum number at fixed value of $\mu_{soft} = 1.1$ GeV and different values of c-quark pole mass.

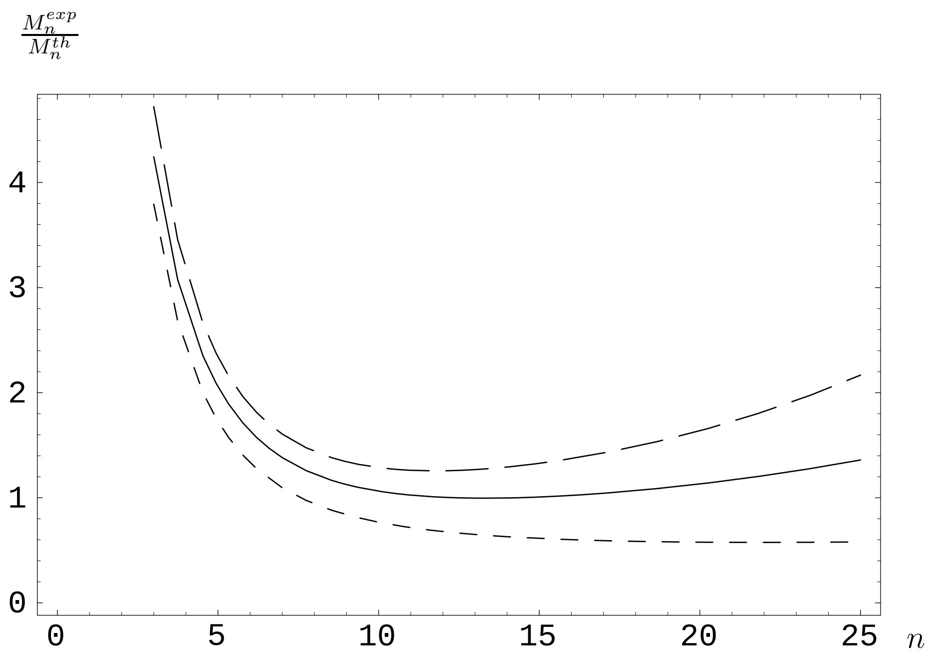

Figure 1: Ratios of experimental and theoretically calculated moments $\frac{M_n^{exp}}{M_n^{th}}$ as functions of momentum number (The solid line is for $M_c^{pole} = 2.03$ GeV, the line with short dashes is for $M_c^{pole} = 1.9$ GeV and the one with long dashes is for $M_c^{pole} = 2.1$ GeV. All ratios are plotted for $\mu_{hard} = 2.0$ GeV and $\mu_{soft} = 1.1$ GeV).

[8]The precise values of other parameters, that is gluon condensate and continuum threshold are not so important, as the correlation function depends on them weakly.

[9]Note, that at NLO, due to vanishing of anomalous dimension of the current under consideration, the dependence of two-point correlator from μ_{hard} enters only through $\alpha_s(\mu_{hard})$. So, the particular value of μ_{hard} is not very important.

The performed analysis shows that to satisfy chosen criteria we need very low values of $\mu_{soft} \approx 1.1$ GeV or lower. On the other hand, as was shown in [32], to match perturbative heavy quark potential with full QCD potential at characteristic distances of quarks bound inside heavy-heavy mesons, we should have the soft scale μ_{soft} in the region $1.3 - 2.0$ GeV[10] To achieve this goal we modify the original TIPT for theoretical moments by shifting the leading order Coulomb potential by a part of constant term in NLO potential, that is

$$V^{(0)}_{modified} = -\frac{C_F \alpha_v}{r} = -\frac{C_F \alpha_s}{r}(1 + t\left(\frac{\alpha_s}{4\pi}\right)) \tag{58}$$

The resulting theoretical moments will now depend not only from hard and soft scales but also from the value of the shift parameter t. To get rid of the latter dependence, while keeping in mind our goal, we perform optimization in parameter t, require the NLO corrections to theoretical moments at given momentum number n do not exceed 1%[11]. As a result we get $M_c^{pole} = 1.96 \pm 0.05$ GeV. At Fig. 1 we show the ratios of experimental to theoretical moments as functions of momentum number at fixed value of $\mu_{soft} = 1.3$ GeV and different values of c-quark pole mass. We see, that within the error bars our results for the c-quark pole mass from these two estimates agree with each other as well as with the results of other estimates [32].

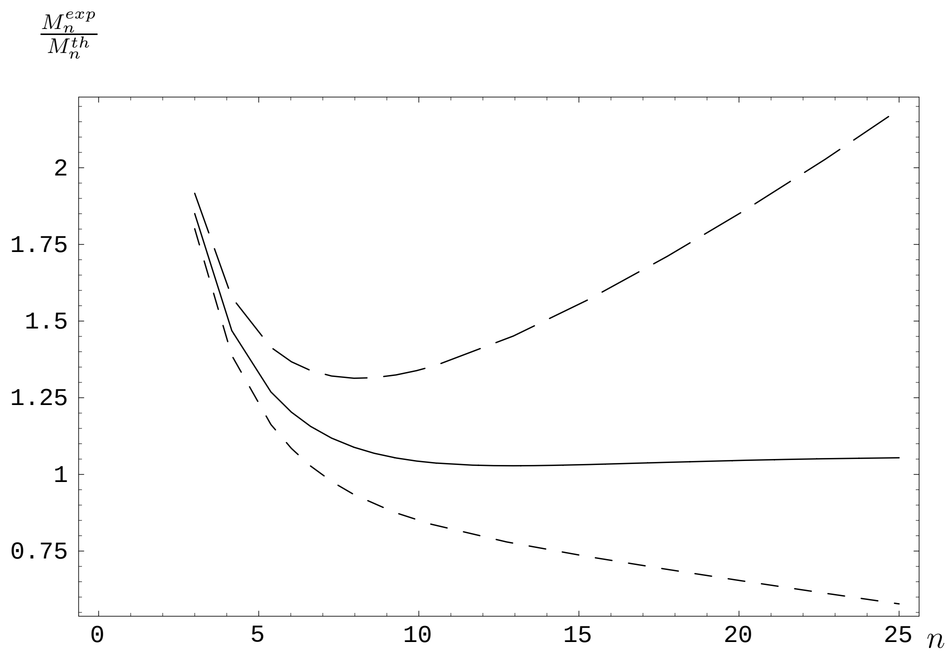

Figure 2: Ratios of experimental and theoretically calculated moments $\frac{M_n^{exp}}{M_n^{th}}$ as functions of momentum number in modified TITP (The solid line is for $M_c^{pole} = 1.96$ GeV, the line with short dashes is for $M_c^{pole} = 1.85$ GeV and the one with long dashes is for $M_c^{pole} = 2.05$ GeV. All ratios are plotted for $\mu_{hard} = 2.0$ GeV and $\mu_{soft} = 1.3$ GeV).

[10] Such choice of soft scale is also desirable in order to have reliable perturbative predictions for theoretical moments.
[11] Note, that here we will need already $\alpha_v(\mu_{soft}) = 0.43 - 0.48$

Having extracted the value of c-quark pole mass from B_c-sum rules, an analogous analysis can be performed for other definitions of heavy quark masses. Below we give the numerical values of these masses obtained in the same way and comment on their relation with each other. First, let us consider the running c-quark mass related to the pole one by the following relation [34, 35]

$$\begin{aligned}\frac{M^{pole}}{\bar{m}(\bar{m})} &= 1+1.333\left(\frac{\alpha_s(\bar{m})}{\pi}\right)+[13.44-1.041n_l]\left(\frac{\alpha_s(\bar{m})}{\pi}\right)^2 \\ &\quad +[190.1-26.7n_l+0.653n_l^2]\left(\frac{\alpha_s(\bar{m})}{\pi}\right)^3 \end{aligned} \tag{59}$$

Our estimates for the $\bar{m}(\bar{m})$ mass are

$$\bar{m}_c(\bar{m}_c) = 1.40 \pm 0.07\ GeV \tag{60}$$

However, not pole not running quark masses are qood ones when one works in the region of energies close to threshold. As we have seen in the analysis performed above the two-point correlation function has a strong dependence on renormalization scales as well as correlation between the pole mass and strong coupling constant (the situation is similar for running quark masses), so to reduce this dependence somewhat it's would be more appropriate to explore quark masses, whose use can eliminate as much as possible all such dependencies and correlations among parameters. For this reason here we give estimates of $1S$ and potential subtracted c-quark masses. The former can be related to the pole mass M^{pole} with the help of following formulae [33]

$$M^{1S} = M^{pole}[1-\triangle^{LO}-\triangle^{LO}\delta^1-\triangle^{LO}\delta^2] \tag{61}$$

where

$$\triangle^{LO} = \frac{C_F^2\alpha_s^2}{8}, \tag{62}$$

$$\delta^1 = \left(\frac{\alpha_s}{\pi}\right)[\beta_0(L+1)+\frac{a_1}{2}], \tag{63}$$

$$\begin{aligned}\delta^2 &= \left(\frac{\alpha_s}{\pi}\right)^2[\beta_0^2(\frac{3}{4}L^2+L+\frac{\zeta_3}{2}+\frac{\pi^2}{24}+\frac{1}{4})+\beta_0\frac{a_1}{2}(\frac{3}{2}L+1)+\frac{\beta_1}{4}(L+1) \\ &\quad +\frac{a_1^2}{16}+\frac{a_2}{8}+(C_A-\frac{C_F}{48})C_F\pi^2],\end{aligned} \tag{64}$$

$$L \equiv \ln(\frac{\mu}{C_F\alpha_s M^{pole}}), \tag{65}$$

and

$$\begin{aligned}a_2 &= \left(\frac{4343}{162}+4\pi^2-\frac{\pi^4}{4}+\frac{22}{3}\zeta_3\right)C_A^2-\left(\frac{1798}{81}+\frac{56}{3}\zeta_3\right)C_ATn_l \\ &\quad -\left(\frac{55}{3}-16\zeta_3\right)C_FTn_l+\left(\frac{20}{9}Tn_l\right)^2\end{aligned} \tag{66}$$

With the use of these formulae we have the following estimate for $1S$ c-quark mass at optimized value of $\mu_{soft}=1.3$ GeV[12]

$$M^{1S} = 1.49 \pm 0.07\ \text{GeV} \tag{67}$$

[12]This value is taken from the previous analysis for c-quark pole mass performed in modified TIPT, as the shift of leading order Coulomb potential implicitly includes part of NNLO corrections. However, as we do not have up to date the complete NNLO order analysis this result should be taken with care.

Having done this estimate it is instructive to compare it with the estimates of NLO analysis done in 1S scheme. To obtain the expressions for theoretical moments in this scheme we make a following substitution

$$\frac{1}{(m_b+m_c)^{2n}} \to \frac{1}{(M_b^{1S}+M_c^{1S})^{2n}} \exp^{-2n\triangle^{LO}(\alpha_s)}\{1-2n\frac{M_b^{1S}}{M_b^{1S}+M_c^{1S}}\triangle^{LO}(\alpha_s)\delta^1(M_b^{1S},\alpha_s,\mu_{soft}) - 2n\frac{M_c^{1S}}{M_b^{1S}+M_c^{1S}}\triangle^{LO}(\alpha_s)\delta^1(M_c^{1S},\alpha_s,\mu_{soft})\} \quad (68)$$

In all other places the heavy quark pole masses should be changed to $1S$ masses. Varying the scales in the same ranges as we did for pole c-quark mass we obtain, that the sum rules are stable themselves, but the ratio of experimental to theoretical moments is far from unity for reasonable values of $1S$ c-quark mass[13]. To solve this problem we again consider the modified TIPT for theoretical moments. It is important, when writing expressions for theoretical moments, to make for consistency a similar shift in NLO order relation between pole and $1S$ masses. Performing calculations in modified $1S$ scheme we obtain the following estimate for $1S$ c-quark mass: $M_c^{1S} = 1.52 \pm 0.05$ GeV, which, within the errors, is in agreement with previous estimate for $1S$ c-quark mass. Fig. 3 shows us the ratios of experimental to theoretical moments as functions of momentum number at fixed values of $\mu_{soft} = 1.3$ GeV and $1S$ c-quark mass.

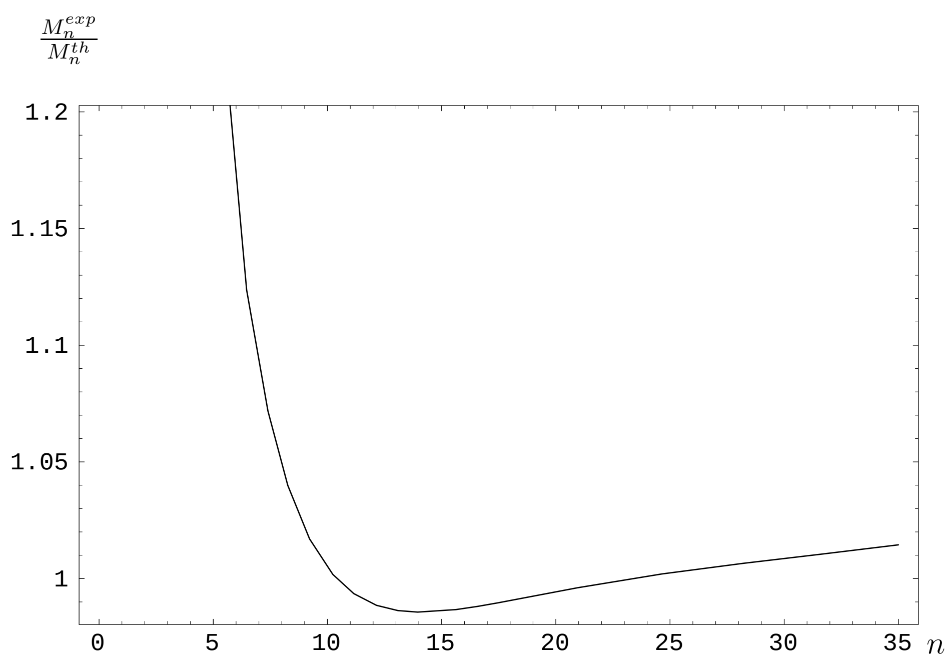

Figure 3: Ratio of experimental and theoretically calculated moments $\frac{M_n^{exp}}{M_n^{th}}$ as functions of momentum number in modified TITP ($M_c^{1S} = 1.52$ GeV, $\mu_{hard} = 2.0$ GeV and $\mu_{soft} = 1.3$ GeV).

[13]Recall that this mass by definition differs from a half of J/Ψ mass on a small value given by nonperturbative corrections.

To finish the discussion of $1S$ c-quark mass we note, that even this mass was defined as M_c^{pole} minus one half of perturbative Coulomb energy in J/Ψ-meson, it can be equally well applied for B_c-meson as well as other systems, containing c-quark. The arguments here are the analog of Upsilon expansion for charmed quarks and the fact, that the Coulomb energy for static quarks does not depend on their flavors.

The estimate for the value of potential subtracted c-quark mass was obtained from the relation of the latter to the running quark mass

$$\begin{aligned} m_{PS}(\mu_f) &= \bar{m}(\bar{m})\{1 + \frac{4\alpha_s(\bar{m})}{3\pi}[1 - \frac{\mu_f}{\bar{m}(\bar{m})}] + \left(\frac{\alpha_s(\bar{m})}{\pi}\right)^2 [13.44 - 1.04 n_l \\ &\quad - \frac{\mu_f}{3\bar{m}(\bar{m})}(a_1 + 4\pi\beta_0[\ln\frac{\mu_f^2}{(M^{pole})^2} - 2])]\} \end{aligned} \tag{69}$$

Thus, at $\mu_f = 1.5$ GeV we have

$$m_{PS}(1.5\mathrm{GeV}) = 1.42 \pm 0.07\mathrm{GeV} \tag{70}$$

Having completed the estimates of different c-quark masses, we can now perform estimates of the mass and coupling constant for the ground B_c-meson state, fixing the heavy quark masses at their central values.

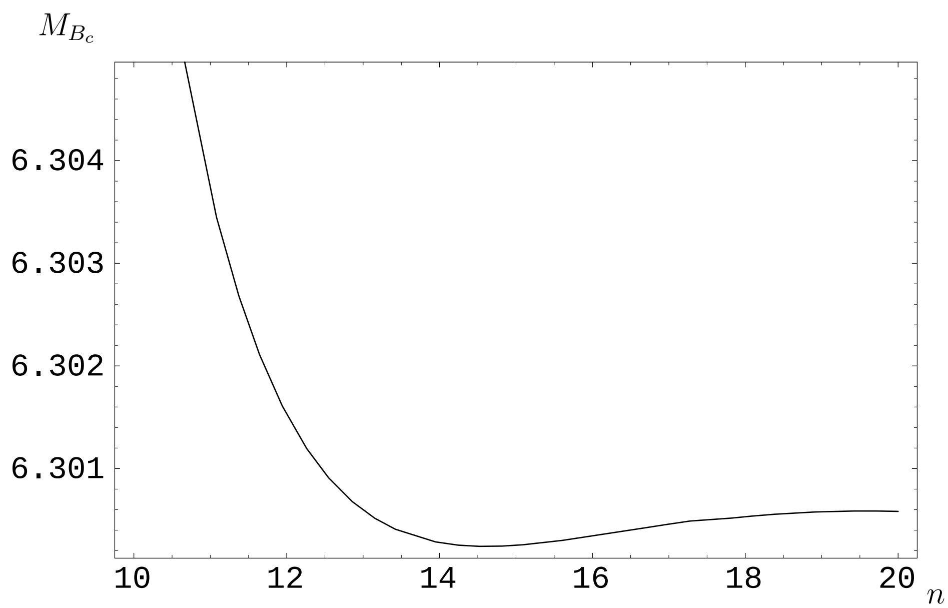

Figure 4: The values of B_c - meson mass extracted from the two-point NRQCD sum rules in moment scheme.

The results for mass and coupling of B_c-meson in momentum scheme of NRQCD sum rules can be easily seen from Fig. 4 and 5.

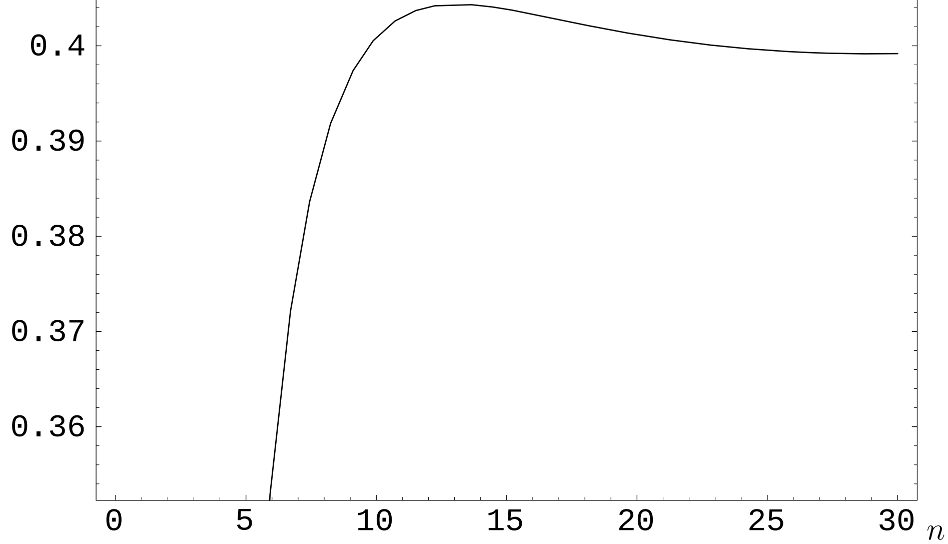

Figure 5: The values of B_c - meson coupling constant extracted from the two-point NRQCD sum rules in moment scheme.

Recently, the estimate of B_c mass was obtained in the perturbative potential approach by fitting the masses of J/Ψ and Υ in order to get a good covergency in α_s and extract the heavy quark masses [36]. The perturbative mass $m_{B_c} = 6.323 \pm 0.007$ GeV is very close to the our estimate in the framework of QCD sum rules. This fact indicates a small nonperturbative correction, which was limited by -60 MeV in [36].

8 Conclusion

In this paper we have presented complete NLO analysis for B_c-meson two-point NRQCD sum rules. Analitycal results for perturbative spectral density and gluon condensate contribution with account for summed Coulomb corrections are derived and analyzed. A detail numerical analysis as well as discussion on the determination of various c-quark masses together with couplings and masses of lowest lying B_c-meson resonances from the mentioned sum rules are provided. The analysis shows that to reduce the uncertainties of calculations it is mandatory to have complete NNLO expressions for theoretical moments, which we plan to accomplish in nearest future.

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Appendix A

In this Appendix we present the two-loop result for perturbative spectral density of B_c meson [23]:

$$\begin{aligned}\rho_{\rm pert}(t) &= \frac{3}{8\pi t}\bar{q}^4 v\Big\{1+\frac{4\alpha_s}{3\pi}\Big\{\frac{3}{8}(7-v^2) \\ &+\sum_{i=b,c}\Big[(v+v^{-1})(L_2(\alpha_1\alpha_2))-L_2(-\alpha_i)-\log\alpha_1\log\beta_i \\ &A_i\log\alpha_i+B_i\log\beta_i\Big]\Big\}+O(\alpha_s^2)\Big\}\end{aligned} \tag{71}$$

where

$$L_2(x) = -\int_0^x \frac{dy}{y}\log(1-y) \tag{72}$$

and

$$\begin{aligned}A_i &= \frac{3}{2}\frac{3m_i+m_j}{m_i+m_j}-\frac{19+2v^2+3v^4}{32v}-\frac{m_i(m_i-m_j)}{\bar{q}^2 v(1+v)}\Big(1+v+\frac{2v}{1+\alpha_i}\Big); \\ B_i &= 2+2\frac{m_i^2-m_j^2}{\bar{q}^2 v}; \\ \alpha_i &= \frac{m_i}{m_j}\frac{1-v}{1+v};\quad \beta_i=\sqrt{1+\alpha_i}\frac{(1+v)^2}{4v}\end{aligned} \tag{73}$$

$$\bar{q}^2 = t-(m_b-m_c)^2;\quad v=\sqrt{1-4\frac{m_b m_c}{\bar{q}^2}} \tag{74}$$

Appendix B

In this Appendix we have collected some formulae, needed for calculation of NRQCD moments in next to leading order.

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{x^{\nu}}e^{xt}dx &= \frac{t^{\nu-1}}{\Gamma(\nu)}, &(75)\\
\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\ln x}{x^{\nu}}e^{xt}dx &= \frac{t^{\nu-1}}{\Gamma(\nu)}[\Psi(\nu)-\ln t], &(76)
\end{aligned}
$$

$$
\begin{aligned}
w_p^0 &= -\frac{1}{p!\Gamma(\frac{p}{2})}\int_0^{\infty}dt\int_0^{\infty}du\frac{1}{(1+t+u)^2}\ln^p\left(\frac{(1+t)(1+u)}{tu}\right) = -\frac{(p+1)\zeta_{p+1}}{\Gamma(\frac{p}{2})}, &(77)\\
w_p^1 &= \frac{1}{p!\Gamma(\frac{p}{2})}\int_0^{\infty}dt\int_0^{\infty}du\frac{1-\ln(1+t+u)}{(1+t+u)^2}\ln^p\left(\frac{(1+t)(1+u)}{tu}\right)\\
&= -\left\{\frac{(1+p)}{\Gamma(\frac{p}{2})}\left[\gamma_E\zeta_{p+1}+\sum_{m=0}^{\infty}\frac{\Psi(2+m)}{(1+m)^{p+1}}\right]+\frac{2}{\Gamma(\frac{p}{2})}\sum_{l=0}^{p-1}\sum_{m=0}^{\infty}(-1)^{p-l}\frac{(1+l)\Psi^{(p-l)}(2+m)}{(p-l)!(1+m)^{1+l}}\right\}, &(78)
\end{aligned}
$$

where ζ_p is the Rieman zeta function for argument p.